CHAPTER ONE

INTRODUCTORY SET THEORY

1. Introduction

In this book we shall be concerned with the study of collections of objects. Each object of a given collection will be called a *point*. A collection of points will be called a *point set* or, more simply, a *set*, and each of the points will be referred to as an element of this set. In any particular discussion the set of all points under consideration will be known as the *universe*. As far as we are concerned in that discussion no other points exist.

To simplify the statements of theorems and problems, we shall always assume, unless otherwise specifically stated, that all the sets mentioned in a single discussion lie in the same universe.

In the following discussion the universe will be denoted by S. This means precisely that, given any point p, p is an element of S. This fact is indicated by the notation $p \in S$. If A is a point set, then, for each point p, exactly one of the following statements is true: (i) p is an element of A (denoted by $p \in A$), or (ii) p is not an element of A (denoted by $p \notin A$). The set of all points p satisfying (i) is called the *subset* A of S. The set of all points p satisfying (ii) is called the *complement* of A in S, or, more simply, the *complement* of A.

Problem 2 shows that the complement of a set depends very strongly on the universe.

Let A and B be subsets of a universe S. We say that "A is contained in B" if and only if every point of A is a point of B. Notationally, "A is contained in B" is written $A \subset B$. The statement "B contains A" (written $B \supset A$) is made if and only if $A \subset B$. Two subsets A and B of a universe S are equal (written $A = B$) if and only if $A \subset B$, and $B \subset A$.

THEOREM 1.1. Let A and B be subsets of a universe S. Then

(*a*) If A is contained in B, then the complement of B is contained in the complement of A.

(*b*) If the complement of B is contained in the complement of A, then A is contained in B.

Proof. (*a*) We must prove that the complement of B is contained in the complement of A. To do this, let p be any point of the complement of B. This means $p \notin B$. But every point of A is a point of B, since $A \subset B$. Thus $p \notin A$; hence p must be a point of the complement of A. We have thus shown that any point of the complement of B is a point of the complement of A, and this is precisely what we mean by the statement "the complement of B is contained in the complement of A."

(*b*) Let $p \in A$. The proof will be complete when we show that $p \in B$. Since $p \in A$, p is not an element of the complement of A; hence p is not an element of the complement of B. Thus $p \in B$.

Theorem 1.1 tells us that either of the conditions "$A \subset B$" and "(complement of B) $\subset$ (complement of A)" implies the other one. For this reason we say that these are equivalent conditions, and that either one of them holds if and only if the other one holds also. We now restate Theorem 1.1 in this "if and only if" language.

THEOREM 1.1. Let A and B be subsets of a universe S. Then $A \subset B$ if and only if (complement of B) $\subset$ (complement of A).

THEOREM 1.2. Let A and B be subsets of a universe S. Then $A = B$ if and only if (complement of A) $=$ (complement of B).

Proof. Since this is an "if and only if" theorem, we must prove two things:

I. If $A = B$, then (complement of A) $=$ (complement of B).

II. If (complement of A) $=$ (complement of B), then $A = B$.

We shall prove I. The proof of II is left as an exercise. Since $A = B$, we know that $A \subset B$; thus (complement of B) $\subset$ (complement of A), by Theorem 1.1. Also, since $A = B$, $B \subset A$; hence (complement of A) $\subset$ (complement of B), again by Theorem 1.1. Thus (complement of A) $=$ (complement of B), and the proof of I is complete.

A subset K of a universe S is defined by dividing the points of S into two classes. The first class consists of all points of S which are not in K, and the set of all these points is called the complement of K. The other class consists of all points of S that are in K. These statements are a repetition of what has been said before. It may not have been noticed, however, that we may pick the first class, namely the complement of K, as the set of all points of S. The set K which is thus uniquely defined contains no points at all. It is called the *null set*, and has as its complement the universe S. We shall use the symbol $\emptyset$ in this text if and only if we mean the null set. The null set is frequently referred to as the empty set.

PROBLEMS

1. Let A be a subset of the universe S, and B the complement of A. Prove that A is the complement of B.

2. (*a*) Let the universe consist of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, and let A be the set consisting of the digits 1, 3, 5. Find the complement of A.
(*b*) Using the same set A, find the complement of A if the universe is the set consisting of the digits 1, 3, 5, 7, 9.

3. Give the details of the proof of II in Theorem 1.2.

4. If $A \subset B$ and $B \subset C$, show that $A \subset C$.

5. Let A be any set. Then $\emptyset \subset A$.

2. Indexed Sets

Let S be a universe. Suppose that we desire to talk about a collection of subsets of S. To distinguish between the sets of this collection it is convenient to give names to them. For this purpose we shall introduce a collection of names Λ (this is the Greek letter capital lambda). The elements of Λ will be denoted by $\alpha, \beta, \cdots$, and will be called the names of our sets. For example, we might have a collection of subsets of S of which certain members were A_α, A_β, A_γ, etc.

The student has seen this idea before. In calculus we deal with sequences of real numbers $(r_1, r_2, r_3, \cdots, r_n, \cdots)$. This means that we single out certain real numbers and give them the names $1, 2, 3, \cdots, n, \cdots$. Thus our collection of names is in this case the collection of all positive integers.

We see that the process of assigning names from the collection Λ is precisely that of assigning a subscript or index to each set considered. It is for this reason that Λ is known as an *index set*, and the process of assigning names from Λ is called *indexing*, the individual names being known as *indices*. The collection of subsets of S to which names have been assigned is said to have been *indexed by* Λ. We shall indicate this collection by the symbol $\{A_\alpha\}_{\alpha\in\Lambda}$ or, more simply, by $\{A_\alpha\}$ where the index set Λ is understood.

Definition 2.1. Let Λ be an index set, S a universe, and $\{A_\alpha\}$ a collection of subsets of S indexed by Λ. Then

(*a*) the union of the sets $\{A_\alpha\}$ is the set denoted by $\bigcup_{\alpha\in\Lambda} A_\alpha$. A point p of S lies in $\bigcup_{\alpha\in\Lambda} A_\alpha$ if and only if p belongs to A_α for at least one α. If Λ is empty then $\bigcup_{\alpha\in\Lambda} A_\alpha = \emptyset$.

(*b*) The intersection of the sets $\{A_\alpha\}$ is the set denoted by $\bigcap_{\alpha\in\Lambda} A_\alpha$. A point p of S lies in $\bigcap_{\alpha\in\Lambda} A_\alpha$ if and only if p belongs to A_α for every α. If Λ is empty then $\bigcap_{\alpha\in\Lambda} A_\alpha = S$.

When only a few subsets of a universe S are considered, we frequently name them with different letters. Thus we interpret $A \cup B = \bigcup_{i\in\{1,2\}} A_i$ where $A_1 = A$, $A_2 = B$. Similarly, $A \cap B = \bigcap_{i\in\{1,2\}} A_i$ where $A_1 = A$, $A_2 = B$.

As an example of the above concepts, consider a universe whose points are the digits 1, 2, 3, 4, 5, 6, 7, 8, 9. Suppose subsets A, B, C of S are defined as follows: A consists of the points 1, 2, 3; B consists of the points 3, 4, 5; and C consists of the points 5, 6, 7. The set $A \cup B$ consists of the points 1, 2, 3, 4, 5 since each of these points belongs to at least one of the sets A, B and no other point of S belongs to either A or B. In the same way the set $A \cup C$ consists of the points 1, 2, 3, 5, 6, 7. The set that is the intersection of $A \cup B$ and $A \cup C$ is denoted by $(A \cup B) \cap (A \cup C)$. This set consists of all points belonging to both the set $(A \cup B)$ and the set $(A \cup C)$: namely, the points 1, 2, 3, 5. The set $A \cup (B \cap C)$ consists also of the points 1, 2, 3, 5. Thus, in this particular example, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. In the next paragraph we shall find out whether or not this is true in general.

Let A, B, C be subsets of a universe S. We shall try to prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. To do this we must show two things: namely, that each of these sets is contained in the other. Let $p \in A \cup (B \cap C)$. Then p belongs to at least one of the sets A and $(B \cap C)$. If $p \in A$, then p belongs to both of the sets $(A \cup B)$ and $(A \cup C)$, and hence to their intersection. If $p \notin A$, then $p \in (B \cap C)$; i.e., p belongs to both the sets B and C. This means again that p is a point of $(A \cup B) \cap (A \cup C)$. We have thus proved that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. It remains to be proved that $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Let $p \in (A \cup B) \cap (A \cup C)$. If $p \in A$, then $p \in A \cup (B \cap C)$. If $p \notin A$, then $p \in B$ and $p \in C$; thus $p \in B \cap C$. It follows that $p \in A \cup (B \cap C)$, and we have proved that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

PROBLEMS

6. Let A and B be subsets of a universe S. Prove
   (*a*) $A \cup B = B \cup A$.
   (*b*) $A \cap B = B \cap A$.
   (*c*) $A \subset A \cup B$, $B \subset A \cup B$.
   (*d*) $A \cap B \subset A$, $A \cap B \subset B$.

(*e*) $A \cup B = A$ if and only if $B \subset A$.
(*f*) $A \cap B = A$ if and only if $A \subset B$.
(*g*) $A \cup \emptyset = A$.
(*h*) $A \cap \emptyset = \emptyset$.
(*i*) $A \cup S = S$.
(*j*) $A \cap S = A$.

7. Let Λ be an index set, S a universe, and $\{A_\alpha\}$ a collection of subsets of S indexed by Λ. Prove

(*a*) $\bigcup_{\alpha \in \Lambda} A_\alpha \supset A_\alpha$ for every $\alpha \in \Lambda$, if $\Lambda \neq \emptyset$.

(*b*) $\bigcap_{\alpha \in \Lambda} A_\alpha \subset A_\alpha$ for every $\alpha \in \Lambda$, if $\Lambda \neq \emptyset$.

(*c*) For any subset B of S, $B \cap \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcap_{\alpha \in \Lambda} (B \cap A_\alpha)$, if $\Lambda \neq \emptyset$.

(*d*) For any subset B of S, $B \cup \bigcup_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} (B \cup A_\alpha)$, if $\Lambda \neq \emptyset$.

8. Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

9. Prove, if $A \cap B = A$ and $A \cup B = A$, then $A = B$.

3. The Theorem of DeMorgan

Let S be a universe and A, B two of its subsets. The set of points that are either in A or in B has been denoted by $A \cup B$. The set of points that are in both A and B has been denoted by $A \cap B$. It is now convenient to introduce a set that will consist of all points of A that are not in B.

DEFINITION 3.1. Let A and B be subsets of a universe S. The set of all points of A that are not in B will be denoted by $A - B$.

We note from this definition that the complement of a subset A of a universe S may be denoted by $S - A$.

THEOREM 3.2 (*DeMorgan's Theorem*). Let Λ be an index set, S a universe, and $\{A_\alpha\}$ a collection of subsets of S indexed by Λ. Then

(*a*) $S - \bigcup_{\alpha \in \Lambda} A_\alpha = \bigcap_{\alpha \in \Lambda} (S - A_\alpha)$.

(*b*) $S - \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} (S - A_\alpha)$.

Proof. If $\Lambda = \emptyset$, both (*a*) and (*b*) are immediate. We thus assume $\Lambda \neq \emptyset$.

(*a*) We must show that each of these sets is contained in the other. This is conveniently done in the following steps.

1. Let $p \in S - \bigcup_{\alpha \in \Lambda} A_\alpha$; then $p \notin \bigcup_{\alpha \in \Lambda} A_\alpha$.

2. Thus p is not in any A_α; hence $p \in S - A_\alpha$ for every α.

3. This means that $p \in \bigcap_{\alpha \in \Lambda} (S - A_\alpha)$; hence $S - \bigcup_{\alpha \in \Lambda} A_\alpha \subset \bigcap_{\alpha \in \Lambda} (S - A_\alpha)$.

4. Let $q \in \bigcap_{\alpha \in \Lambda} (S - A_\alpha)$; then q is not in any A_α.

5. This means $q \in S - \bigcup_{\alpha \in \Lambda} A_\alpha$ so that $\bigcap_{\alpha \in \Lambda} (S - A_\alpha) \subset S - \bigcup_{\alpha \in \Lambda} A_\alpha$, and (*a*) is proved.

(*b*) Define $B_\alpha = S - A_\alpha$. Then, from Problem 1, $S - B_\alpha = S - (S - A_\alpha) = A_\alpha$. Thus, by (*a*), $S - \bigcup_{\alpha \in \Lambda} B_\alpha = \bigcap_{\alpha \in \Lambda} (S - B_\alpha)$, so that $\bigcup_{\alpha \in \Lambda} B_\alpha = S - \bigcap_{\alpha \in \Lambda} (S - B_\alpha)$. But $B_\alpha = S - A_\alpha$; hence $\bigcup_{\alpha \in \Lambda} (S - A_\alpha) = S - \bigcap_{\alpha \in \Lambda} A_\alpha$, and the proof of (*b*) is complete.

We may, of course, apply DeMorgan's theorem to two subsets A and B of a universe S, and obtain the following formulas:

$$S - (A \cup B) = (S - A) \cap (S - B)$$

$$S - (A \cap B) = (S - A) \cup (S - B)$$

The student has noticed the name DeMorgan attached to the previous theorem. The name is given since this theorem is the exact analog of the theorem in logic which is known by the same name. Let us consider briefly the connection between logic and set theory. Let α and β be statements concerning the points of a universe S, such that it can be decided, for each point p of S, whether α is true or false for the point p, and whether β is true or false for the point p. Let A_α denote the set of all points of S for which α is true, and A_β the set of all points of S for which β is true. Consider the statement: "If α is true, then β is true." This is precisely equivalent to the statement: "$A_\alpha \subset A_\beta$." But, by Theorem 1.1, $A_\alpha \subset A_\beta$ if and only if $S - A_\beta \subset S - A_\alpha$; that is, "If β is false, then α is false." This shows that the two statements, "If α is true, then β is true," and "If β is false, then α is false," are equivalent in the sense that, if either has been proved, then the other must also be true. Although they are equivalent in this logical sense, frequently the simplest method of proving the first statement is to show that the second one is true. This is the typical argument by contradiction (reductio ad absurdum).

As another example of the connection between logic and set theory, consider the result of Problem 4, where the sets A, B, C are subsets of S consisting, respectively, of all points for which statements α, β, and γ are true. The statement of Problem 4, namely, "If $A \subset B$ and $B \subset C$, then $A \subset C$," interpreted logically, becomes "If α implies β and β implies γ, then α implies γ."