INTRODUCTION

In recent years it has become increasingly apparent that topology is one of the major branches of modern mathematics. Besides the importance of this subject in its own right, an understanding of it will help the student immeasurably in his courses in advanced calculus, real variables, complex variables, and other courses in analysis. It is therefore essential that students be introduced to this subject at as early a stage as possible in their undergraduate careers. This book is intended as a textbook for such an early introductory course.

A major difficulty in the teaching of advanced undergraduate courses consists in getting the students to follow chains of reasoning containing more than one link. The first two chapters are pointed toward this difficulty. Chapter I starts out rather gently with the elements of set theory, and introduces the concepts of mappings, equivalence, and order relations. It closes with a brief account of finite, infinite, countable, and uncountable sets. Students are usually so intrigued by the infinite that this topic is particularly suitable as a first introduction to a closely argued chain of reasoning. It is also vital to their later work.

Chapter II is devoted to a study of the real number system. In this familiar setting the student is exposed to chains of reasoning typical of many encountered in later chapters. This chapter is fundamental to the remainder of the book. Its purpose is threefold: first, to impart to the student a sound knowledge of real numbers and their fundamental properties; second, to provide a reservoir of examples of topological concepts; third, to furnish a clear motivation for the abstract generalizations of later chapters. In particular, in connection with the third point, we emphasize in Chapter II the fundamental role played by open sets. This motivates clearly the study of general topological spaces in Chapter III.

In Chapter IV we return from these generalizations to a study of metric spaces. This is followed by a long chapter on Arcs and Curves, in which arcs, simple closed curves, and the simple closed surface or two-sphere are characterized. Related material on Peano spaces, cyclic element theory, and property S is also included.

Chapter VI is short, and is devoted to the equivalence of partitionability and property S in metric spaces. This result is basic to Bing's recent characterization of E^3. Chapter VII contains a detailed treatment of alternative forms of the Axiom of Choice, such as Zorn's lemma, and the Tychonoff and well-ordering theorems. Several applications of these alternative forms to topology are given.

Some additional remarks on our organization of the subject matter, as well as some specific comments on certain theorems, may be helpful to the reader. Some of these remarks will also indicate in more detail the content of the text.

In Chapter IV, the section on Metrizability contains a necessary and sufficient condition for a space to be metrizable. This condition is essentially that found in the study of uniform structures. If desired, this can be omitted, since the next section contains the usual metrization theorem for regular, second countable T_1 spaces. Moreover, this latter result is sufficient for the later chapters. Another alternative would be to present the Smirnov metrization theorem (Yu. M. Smirnov, On Metrization of Topological Spaces, *American Mathematical Society*, *Translation Number* 91, 1953) which gives a necessary and sufficient condition of a similar nature to the usual theorem for regular, second countable T_1 spaces.

Included as a problem in the last section of Chapter IV is an outline of some of the main theorems on Moore-Smith sequences, the proofs of which are to be worked out as a short term paper by the student. Other problems of the term paper variety may be found in Chapters IV and V. The proofs of theorems in the text are in no way dependent on the solution of these problems.

Depending upon the background of the students in the class, it may or may not be desirable to follow Chapter IV by Chapter VII on the Axiom of Choice, and then to continue with Chapter V. In any event, no use of the results obtained in Chapter VII is made in any of the earlier chapters, although there are some instances where the arguments could have been shortened by such use.

The long Chapter V on Arcs and Curves contains, as previously remarked, characterizations of the arc, the simple closed curve, and the simple closed surface. Algebraic methods of proof have been avoided to keep the text on as elementary a level as possible. The proofs are long, and we have purposely tried to err on the side of giving too many details rather than too few. For this reason, the student should be required to outline the main steps of the arguments so that he will not fail to see the forest for the trees. Indeed, in general, throughout the text it is necessary that the student not merely follow but actually think through the proofs for himself, in order to gain real understanding.

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The proof we give of the characterization of the simple closed surface is essentially van Kampen's organization of Zippin's proof. By slight modification of details we obtain at the same time the Jordan-Schoenflies theorem.

It will also be noticed that the concepts of components and local compactness are not treated directly until they occur in Chapter V. This was done to avoid endlessly building new machinery in earlier chapters and having it rust before active use of it was made. Indeed, insofar as possible, this has been our general thought concerning the place to introduce new ideas. On the other hand, it is perfectly possible to give many of our results on local compactness and components at some earlier stage deemed appropriate.

One of our reasons for giving so much material on Peano spaces is that they probably provide the richest single source of illustrations of various mathematical techniques used in topology. This remark also reveals our feeling that the most valuable asset a mathematician can have is a knowledge of a wide variety of techniques and methods of proof.

Our classroom experience indicates that it is reasonable to cover Chapters I, II, III, IV, and the first part of Chapter V, in a two-semester course. The experience cited is with undergraduates who have not had advanced calculus. With more advanced students, Chapters I and II could be covered very rapidly, and a considerably greater portion of the text completed.

We have, of course, drawn heavily and freely upon the existing literature in topology, particularly upon the *American Mathematical Society Colloquium Publications* of G. T. Whyburn and R. L. Wilder. Our primary sources of this nature are listed in the bibliographical comments. We also drew heavily upon the rich experience due to the association with and teachings of our various professors. Professor Hall is particularly indebted to Professors G. T. Whyburn, E. J. McShane, J. R. Kline, and J. D. Tamarkin. Professor Spencer is particularly indebted to Professors R. L. Wilder, Gail S. Young Jr., Hans Samelson, and T. H. Hildebrandt. Our classroom experience with the manuscript has also been most helpful, and we are indebted to our students for suggestions they have made.

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Preface To The 2018 Edition

Point set topology has gotten a really bad rap over the last few decades and it pisses me off.

Seriously.

And this is quite personal for me because in a lot of ways, it was point set topology that lead me to my pursuit of mathematics from chemistry and medicine. After I nearly had an aneurysm taking 6 courses in both mathematics and chemistry while caring for my cancer riddled father, I spent my recovery period thinking about whatever information that had successfully penetrated the fog of adrenal anxiety that nearly destroyed me. One fact particularly stood out to me that I realized had puzzled me in both vector and advanced calculus: One course had used open balls to define limits and convergence in $\mathbb{R}^3$ and the latter used open boxes. I quite naturally asked what the difference was. I was told by one teacher to stop asking stupid questions and the other simply ignored my question and went on. It was frustrating, but it solidified my resolve to get to the bottom of all the questions on calculus that my professors either refused to answer or told me it was too advanced for the course.

Analysis courses often supplied answers to *what* and *how* questions about calculus, such as how to rigorously compute limits and prove which functions are Riemann integrable. Sometimes I got partial answers to my deep *why* questions that puzzled me in calculus, such as why the intermediate value theorem was true.

But more often than not, a *complete* answer to most of my *why* questions could be found in point-set topology. The open boxes vs. open balls on $\mathbb{R}^2$ (or $\mathbb{R}^3$) , of course, is a perfect example that now appears totally obvious to me now thanks to the concept of equivalent bases for a topology □ on a set X. These 2 families of open sets on $\mathbb{R}^2$ (or $\mathbb{R}^3$) are equivalent bases that both generate the usual topology of open sets on $\mathbb{R}^2$ (or $\mathbb{R}^3$). Armed with point set topology, the diligent student can also see-with some totally worthwhile effort- that the intermediate value theorem on $\mathbb{R}$ (and by simple extension, $\mathbb{R}^n$) is a trivial consequence of the fact these spaces are connected.

This is a subject that, to me, is to mathematics what chipped beef is to culinary circles: an old standby that in earlier times, was much loved and cherished, but that due to a combination of changing values, a perceived "blandness" compared with other, flashier and more versatile dishes, as well as poor execution of late, has fallen out of favor and is spoken of only in the most derogatory of terms.

This to me is a very sad state of affairs. For many students of mathematics, myself in particular, point-set topology was among my first exposures to truly sophisticated mathematics. It is a subject whose very simplicity gives it a power and beauty that still fascinates me. It is amazing how many deep results one can prove armed simply with basic set theory and a cursory understanding of analysis on the real line. And, of course, a deep understanding of analysis simply isn't possible without a fairly good grasp of compactness, connectedness, open and closed sets and related notions. You *can* study analysis without it, of course, and in the current climate many students do — but the resulting construct is ridiculously complicated; one bemoans the lack of the language of point set topology, which would simplify it and provide an organizing framework for the many different concepts.

I first learned the bare essentials of this beautiful subject as an undergraduate from my mentor, Nick Metas. I later sat in on the upper level course on the subject taught by Gerald Itzkowitz from his deep and beautiful notes and finally mastered the subject in the graduate course of John Terilla. I consider myself doubly blessed in this regard, since I learned the analytic aspects of the subject, such as topological groups and generalized convergence from Itzkowitz and the more geometric aspects such as quotient spaces and homotopy, from Terilla. While Terilla's course probably prepared me for a graduate algebraic topology