CHAPTER XI

TRIGONOMETRIC SERIES AND SERIES OF ORTHOGONAL FUNCTIONS

151. Schwarz's inequality. Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be two sets of n real numbers each. From the fact that the quadratic form

$$\sum_{i=1}^{n} (\lambda x_i + \mu y_i)^2$$

is greater than, or equal to, zero for all real values of λ and μ, we have immediately

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leqq \sum x_i^2 \sum y_i^2. \tag{1}$$

This important formula is known as *Schwarz's inequality.*

In the same order of ideas the inequality

$$\int_a^b [\lambda f(x) + \mu g(x)]^2 dx \geqq 0,$$

or

$$\lambda^2 \int_a^b \overline{f(x)}^2 dx + 2\lambda\mu \int_a^b f(x)\cdot g(x) dx + \mu^2 \int_a^b \overline{g(x)}^2 dx \geqq 0,$$

where $f(x)$ and $g(x)$ are integrable functions and λ and μ are independent of x, leads to the formula

$$\left[\int_a^b f(x)\cdot g(x) dx\right]^2 \leqq \int_a^b \overline{f(x)}^2 dx \int_a^b \overline{g(x)}^2 dx. \tag{2}$$

152. Definition. A series of the form

$$S_m(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{m} (a_n \cos nx + b_n \sin nx),$$

where a_n and b_n are real constants, is called a *trigonometric series.* If m is finite the series is said to have $m + 1$ terms.

Every such series is a continuous function of x with the period 2π. But not every such function of x can be represented as a finite trigonometric series. Such a representation can in general be only approximate. In deciding which of all such representations with a given number of terms we shall consider the best one, we have to give effect to the difference $f(x) - S_m(x)$ for every value of x in an interval of length 2π, say the interval $(0, 2\pi)$. This consideration suggests that we take the integral $\int_0^{2\pi}[f(x) - S_m(x)]dx$ as a measure of this approximation. But this integral might be small in absolute value, or even zero, in cases where the difference $f(x) - S_m(x)$ is fairly large throughout a part of the interval, if it is nearly as large in absolute value and of opposite sign in other parts of the interval. In order to avoid this difficulty we agree to say that that trigonometric series $S_m(x)$ gives the best possible approximate representation of $f(x)$ in the interval $(0, 2\pi)$ of all trigonometric series with the same number of terms if it makes the integral

$$I_m = \int_0^{2\pi}[f(x) - S_m(x)]^2dx \tag{3}$$

a minimum.

We have first to consider whether such a best approximate representation exists. Can the coefficients a_n and b_n be so determined as to make I_m a minimum? A necessary condition for this is that all the first partial derivatives $\frac{\partial I_m}{\partial a_n}$ and $\frac{\partial I_m}{\partial b_n}$ $(n = 0, 1, 2, \cdots, m)$ shall equal zero. Now

$$\frac{\partial I_m}{\partial a_n} = -2\int_0^{2\pi} f(x)\cdot\cos nx dx + 2\int_0^{2\pi} S_m(x)\cdot\cos nx dx$$

and

$$\frac{\partial I_m}{\partial b_n} = -2\int_0^{2\pi} f(x)\sin nx dx + 2\int_0^{2\pi} S_m(x)\sin nx dx$$

for $n \neq 0$, and

$$\frac{\partial I_m}{\partial a_0} = -\int_0^{2\pi} f(x)dx + \int_0^{2\pi} S_m(x)dx.$$

Moreover

$$\left.\begin{aligned}
&\int_0^{2\pi}\cos nx dx = \begin{matrix}0 \ (n \neq 0) \\ 2\pi\ (n = 0)\end{matrix}, \quad \int_0^{2\pi}\sin nx dx = 0 \\
&\int_0^{2\pi}\cos^2 nx dx = \int_0^{2\pi}\sin^2 nx dx = \pi \quad (n \neq 0) \\
&\int_0^{2\pi}\sin mx \cos nx dx = 0 \\
&\int_0^{2\pi}\sin mx \sin nx dx = 0 \quad (m \neq n) \\
&\int_0^{2\pi}\cos mx \cos nx dx = 0 \quad (m \neq n)
\end{aligned}\right\} \quad (4)$$

Hence

$$\left.\begin{aligned}
\frac{\partial I_m}{\partial a_0} &= -\int_0^{2\pi} f(x)dx + \pi a_0 \\
\frac{\partial I_m}{\partial a_n} &= -2\int_0^{2\pi} f(x)\cos nx dx + 2\pi a_n \quad (n \neq 0) \\
\frac{\partial I_m}{\partial b_n} &= -2\int_0^{2\pi} f(x)\sin nx dx + 2\pi b_n
\end{aligned}\right\} \quad (5)$$

In order that these partial derivatives shall equal zero we must have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx.
\end{aligned} \quad (6)$$

It is clear from formulae (5) that the second partial derivative of I_m with respect to a_0 is π, and with respect to any one of the other coefficients is 2π, while all the mixed second partial derivatives are zero. Hence if we denote by a_0, a_n, b_n the values of the coefficients given by (6) and by $\bar{a}_0$, $\bar{a}_n$, $\bar{b}_n$ any other values, we have

$$I_m(\bar{a}, \bar{b}) = I_m(a, b) + \pi\left[\frac{(\bar{a}_0 - a_0)^2}{2} + \sum_{n=1}^{m}\{(\bar{a}_n - a_n)^2 + (\bar{b}_n - b_n)^2\}\right].$$

It follows that $I_m(a, b)$ is the minimum value of I_m the existence of which has been in question.

The coefficients given by (6) are called the *Fourier constants* of the function. If we substitute them in $S_m(x)$ we obtain the formula

$$\int_0^{2\pi} f(x)S_m(x)dx = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{m}(a_n^2 + b_n^2)\right]$$
$$= \int_0^{2\pi}[S_m(x)]^2dx.$$

From this it follows that

$$I_m(a, b) = \int_0^{2\pi}\overline{f(x)}^2dx - \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{m}(a_n^2 + b_n^2)\right]. \quad (7)$$

It is clear from (7) that the series $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges since $I_m(a, b) \geqq 0$, and that therefore the series $\sum a_n^2$ and $\sum b_n^2$ converge.

Since the integral in the right member of (7) is independent of m, and since the series in this right member has no negative terms, we have

$$I_{m+1} \leqq I_m.$$

That is, we get better and better approximations by increasing the value of m, unless all the coefficients from a certain point on are zero, in which case $I_{m+1} = I_m$. This raises the question as to whether the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges to $f(x)$. Whatever the answer to this question may be, the series is called the *Fourier series* of the function $f(x)$.

We have been assuming that the function $f(x)$ is continuous in the interval $(0, 2\pi)$, and of period 2π. But only the values of the function in this interval have been considered. The restriction that $f(x)$ be periodic has therefore been unnecessary up to this point. We shall use it however in § 153. And we did not need to assume continuity—it was only necessary to assume that $f(x)$ is integrable in the interval $(0, 2\pi)$, inasmuch as the integrability of $\overline{f(x)}^2$, $f(x) \cos nx$, and $f(x) \sin nx$ follows from this assumption. It is no limitation to confine our attention to an interval of length 2π. For if the interval is (a, b) the substitution $x' = \dfrac{2\pi(x-a)}{b-a}$ replaces it by the interval $(0, 2\pi)$.

153. If in formulae (6) we replace the variable of integration by α and then substitute the values of these constants in the expression for $S_m(x)$, we get

$$S_m(x) = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\left[\frac{1}{2} + \cos(\alpha - x) + \cdots + \cos m(\alpha - x)\right]d\alpha.$$

But

$$\begin{aligned}\sin\frac{\theta}{2}&\left(\frac{1}{2} + \cos\theta + \cdots + \cos m\theta\right)\\ &= \frac{1}{2}\sin\frac{\theta}{2} + \sum_{n=1}^{m}\frac{1}{2}\left[\sin\left(n+\frac{1}{2}\right)\theta - \sin\left(n-\frac{1}{2}\right)\theta\right]\\ &= \frac{\sin\left(m+\frac{1}{2}\right)\theta}{2},\end{aligned}$$

and therefore

$$\frac{1}{2} + \cos\theta + \cdots + \cos m\theta = \frac{\sin\left(m+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

Then

$$S_m(x) = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\frac{\sin\left(m+\frac{1}{2}\right)(\alpha - x)}{2\sin\frac{1}{2}(\alpha - x)}\,d\alpha.$$

If we put $\alpha = x + t$,

$$S_m(x) = \frac{1}{\pi}\int_{-x}^{2\pi - x} f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\,dt,$$

since the integrand is of period 2π. Now

$$\frac{\sin\left(m+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} = \frac{\sin\frac{t}{2}\sin\left(m+\frac{1}{2}\right)t}{2\sin^2\frac{1}{2}t}$$

$$= \frac{\cos mt - \cos(m+1)t}{4\sin^2\frac{1}{2}t}.$$

Hence

$$S_m(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+t)\frac{\cos mt - \cos(m+1)t}{2\sin^2\frac{1}{2}t}\,dt.$$

If we put $\Sigma_m(x) = \dfrac{S_0(x) + S_1(x) + \cdots + S_m(x)}{m+1}$, we have

$$\Sigma_m(x) = \frac{1}{2\pi(m+1)}\int_0^{2\pi} f(x+t)\frac{1-\cos(m+1)t}{2\sin^2\frac{1}{2}t}\,dt$$

$$= \frac{1}{2\pi(m+1)}\int_0^{2\pi} f(x+t)\left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt. \quad (8)$$

Our first problem now is to determine the behavior of $\Sigma_m(x)$ as m increases without limit. If in formula (8) we take $f(x) = 1$, we get

$$1 = \frac{1}{2\pi(m+1)}\int_0^{2\pi}\left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt, \tag{9}$$

since in this case $a_0 = 2$ and $a_n = b_n = 0$ when $n > 0$; and therefore $S_m(x) = \sum_m(x) = 1$ for all values of m. Hence, for the $f(t)$ of formula (8),

$$\sum\nolimits_m(x) - f(x)$$
$$= \frac{1}{2\pi(m+1)}\int_0^{2\pi}[f(x+t) - f(x)]\left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt. \tag{10}$$

We shall for the present assume that $f(x)$ is continuous in the closed interval $(0, 2\pi)$ and therefore uniformly continuous. There is then a positive δ associated with every positive ϵ such that for every x in the interval and every t that is less than δ in absolute value we have

$$|f(x+t) - f(x)| < \epsilon.$$

For $x > 0$, $x + t$ becomes greater than 2π as t varies from 0 to 2π. It is necessary therefore that $f(x)$ be defined for values of x outside of the interval $(0, 2\pi)$. This is provided for by the assumption that $f(x)$ has the period 2π, or that $f(x + 2\pi) = f(x)$. In order that it be continuous it is necessary that it satisfy the condition $f(0) = f(2\pi)$. Now

$$\int_0^{2\pi}[f(x+t) - f(x)]\left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt$$
$$= \int_0^{\delta} + \int_{\delta}^{2\pi-\delta} + \int_{2\pi-\delta}^{2\pi}. \tag{11}$$

If we denote the common integrand in (11) by $\varphi(t)$, and if

$|f(x)| < M$ in the interval $(0, 2\pi)$, we have

$$\left|\frac{1}{2\pi(m+1)}\int_0^\delta \varphi(t)dt\right| \leqq \frac{\epsilon}{2\pi(m+1)}\int_0^\delta \left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt$$

$$< \frac{\epsilon}{2\pi(m+1)}\int_0^{2\pi} \left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt = \epsilon$$

by (9). Similarly we get

$$\left|\frac{1}{2\pi(m+1)}\int_{2\pi-\delta}^{2\pi} \varphi(t)dt\right|$$

$$\leqq \frac{\epsilon}{2\pi(m+1)}\int_{2\pi-\delta}^{2\pi} \left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt$$

$$< \frac{\epsilon}{2\pi(m+1)}\int_0^{2\pi} \left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt = \epsilon.$$

Also

$$\left|\frac{1}{2\pi(m+1)}\int_\delta^{2\pi-\delta} \varphi(t)dt\right|$$

$$< \frac{M}{\pi(m+1)}\int_\delta^{2\pi-\delta} \left[\frac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right]^2 dt$$

$$< \frac{M}{\pi(m+1)\sin^2\frac{1}{2}\delta}\int_\delta^{2\pi-\delta} dt < \frac{2M}{(m+1)\sin^2\frac{1}{2}\delta}.$$

In other words,

$$|\Sigma_m(x) - f(x)| < 2\epsilon + \frac{2M}{(m+1)\sin^2\frac{1}{2}\delta},$$

But for all sufficiently large values of m we shall have

$$\frac{2M}{(m+1)\sin^2\frac{1}{2}\delta} < \epsilon,$$

and therefore

$$|\Sigma_m(x) - f(x)| < 3\epsilon$$

for every x in $(0, 2\pi)$.

We have now proved that if $f(x)$ is continuous in the closed interval $(0, 2\pi)$ with $f(x) = f(x + 2\pi)$ then $\Sigma_m(x) \to f(x)$ as $m \to \infty$. Moreover this approach is uniform in this interval. It follows that

$$\int_0^{2\pi} [f(x) - \Sigma_m(x)]^2 dx \to 0$$

as $m \to \infty$. But we have seen (§ 152) that for a given m

$$\int_0^{2\pi} [f(x) - S_m(x)]^2 dx \leqq \int_0^{2\pi} [f(x) - \Sigma_m(x)]^2 dx,$$

since $\Sigma_m(x)$ is a trigonometric series with $m + 1$ terms. Hence by virtue of (7)

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi}\int_0^{2\pi} \overline{f(x)}^2 dx. \tag{12}$$

154. Equation (12) has been established on the assumption that $f(x)$ is continuous for $0 \leqq x \leqq 2\pi$, with $f(x) = f(x + 2\pi)$. Suppose now that we have a function $g(x)$ which has at most a finite number of discontinuities $\alpha_1, \alpha_2, \cdots, \alpha_k$ in this interval and is of period 2π. These discontinuities are to be such that the limits $g(\alpha_i - 0)$ and $g(\alpha_i + 0)$ exist for $i = 1, 2, \cdots, k$; that is, they are to be of the *first kind.* We shall take $\frac{g(\alpha_i + 0) + g(\alpha_i - 0)}{2}$ as the value of the function at α_i; and $\frac{g(+0) + g(2\pi - 0)}{2}$ as the value of $g(0)$ and of $g(2\pi)$, if such discontinuities occur at the ends of the interval. If we connect the points on the curve $y = g(x)$ whose abscissae are $\alpha_i - \delta$ and $\alpha_i + \delta$, where $\delta > 0$, by a straight line and replace the curve in each interval by the corresponding line, we obtain the graph of a continuous function $f(x)$ with period 2π, it being understood that if such discontinuities occur at the ends of the interval we draw straight lines from $[0, g(0)]$ to $[\delta, g(\delta)]$ and from $[2\pi - \delta, g(2\pi - \delta)]$ to $[2\pi, g(2\pi)]$. If a_n and b_n are the Fourier constants of $f(x)$ we know from § 153 that

$$\int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right]^2 dx$$

can be made as small as we please by a suitable choice of n. Now

$$\begin{aligned}
M' &= \int_0^{2\pi} \left[g(x) - \frac{a_0}{2} - \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right]^2 dx \\
&= \int_0^{2\pi} \left[g(x) - f(x) + f(x) - \frac{a_0}{2} \right. \\
&\qquad\qquad \left. - \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right]^2 dx \\
&= \int_0^{2\pi} [g(x) - f(x)]^2 dx \\
&\quad + \int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right]^2 dx \\
&\quad + 2\int_0^{2\pi} [g(x) - f(x)] \\
&\qquad \cdot \left[f(x) - \frac{a_0}{2} - \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right] dx.
\end{aligned}$$

By virtue of (151) the absolute value of the last integral in the right member of this equation does not exceed the square root of the product of the other two integrals. By taking δ sufficiently near to zero we can make $\int_0^{2\pi} [g(x) - f(x)]^2 dx$ less than any pre-assigned positive number, and we have just seen that

$$\int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right]^2 dx$$

can be made arbitrarily near to zero by taking n sufficiently large. Hence we can make M' as near to zero as we wish. But if c_ν and d_ν are the Fourier constants of $g(x)$ we know from § 152 that

$$\int_0^{2\pi}\left[g(x) - \frac{c_0}{2} - \sum_{\nu=1}^{n}(c_\nu \cos \nu x + d_\nu \sin \nu x)\right]^2 dx \leqq M'.$$

We conclude from this that

$$\frac{c_0^2}{2} + \sum_{\nu=1}^{n}(c_\nu^2 + d_\nu^2) \to \frac{1}{\pi}\int_0^{2\pi}\overline{g(x)}^2 dx \quad \text{as} \quad n \to \infty.$$

155. We are now in a position to prove that if $f(x)$ is continuous in $(0, 2\pi)$ with $f(x) = f(x + 2\pi)$, and if it has a derivative which is continuous except for a finite number of discontinuities of the first kind, it is the sum of its Fourier series.

We denote the Fourier constants of $f(x)$ by a_n and b_n, and those of $f'(x)$ by a_n' and b_n'. Integration by parts gives us for $n > 0$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$
$$= -\frac{1}{n\pi}\int_0^{2\pi} f'(x)\sin nx dx = -\frac{b_n'}{n},$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{n\pi}\int_0^{2\pi} f'(x)\cos nx dx = \frac{a_n'}{n}.$$

Now the series $\sum a_\nu'^2$ and $\sum b_\nu'^2$ converge (§ 152), and moreover, by Schwarz's inequality,

$$\left(\sum_{\nu=m}^{n}|a_\nu \cos \nu x|\right)^2 \leqq \left(\sum_{\nu=m}^{n}\frac{|b_\nu'|}{\nu}\right)^2 \leqq \sum_{\nu=1}^{n} b_\nu'^2 \sum_{\nu=m}^{n}\frac{1}{\nu^2},$$
$$\left(\sum_{\nu=m}^{n}|b_\nu \sin \nu x|\right)^2 \leqq \left(\sum_{\nu=m}^{n}\frac{|a_\nu'|}{\nu}\right)^2 \leqq \sum_{\nu=1}^{n} a_\nu'^2 \sum_{\nu=m}^{n}\frac{1}{\nu^2}.$$

Since $\sum \frac{1}{\nu^2}$ converges it follows that the series

$$\frac{a_0}{2} + \sum(a_\nu \cos \nu x + b_\nu \sin \nu x)$$